LETTERS TO THE EDITOR

COMMENTS ON "ESTIMATING THE VELOCITY PROFILE AND ACOUSTICAL QUANTITIES OF A HARMONICALLY VIBRATING LOUDSPEAKER MEMBRANE FROM ON-AXIS PRESSURE DATA"*

I would like comment on the above paper.[1] While I basically do not disagree with any of it, I believe that the authors may have missed the bigger picture in which this technique lies, as well as some prior art that should be acknowledged.[2]

In its most fundamental form the loudspeaker membrane can be thought of as being a vibrating object, which can be described in a modal manner. This is consistent with the authors' eq. (4). However, there are a number possible functional expansions that can be used. In our book, *Audio Transducers*,[3] we used Bessel functions and cosines as the expansion functions, but others are just as acceptable. In the general case, if we measure N field points then we can find the first N modes through some form of transform in the space. The problem at hand then becomes how to define the transform in a convenient manner. In the subject paper this was done from the Zernike polynomials to axial field points, whereas in our book this was done from the Bessel and cosine representations to far-field polar points in θ and φ. The authors' fig. 6 is identical to our fig. 4-4. Either representation works, but there might be some practicalities involved that favor one choice over the other.

Using the authors' technique one must obtain axial pressure data at a number of radial locations. While not difficult, these measurements are nonetheless not typically done. On the other hand it is very common to measure the full polar field response from a loudspeaker at a fixed radial distance. In other words, the data required to reassemble the cone's motion are already available in the form of the polar response if the expansion functions are chosen as described.

Once one has the required data, exactly the same solution techniques are used in all cases, namely, a singular value decomposition (SVD) of the data to find the significant eigenvalues. (The matrices involved tend to be nearly singular for small values of ka.) The authors describe this well in their sec. 5.1.

Regarding the authors' "practical issues," we investigated some of these. The choice of measurement points is not critical if more points are used than modes required. Some care must be exercised to ensure that all of the radiation modes have been covered. For example, if one only takes measurements along a particular value of φ, then it would not be possible to determine the functional variations in φ [that is, cosine ($n\varphi$)].

The conditions of the linear system will always degrade at low ka values as the matrices become ever more singular. But With SVD this is usually not an issue, except at very low ka values, where there is basically only a single mode possible and very ill-conditioned matrices can result. Reconstruction at these values of ka is usually not required since the diaphragm is generally a rigid body.

The influence of ka is, in fact, quite pronounced. The coefficients of the modal expansion have a strong dependence on ka, and hence this expansion must be done independently at every value of ka. An efficient technique for dealing with this complexity, along with the details of the polar technique, are shown in our U.S. patent application # 20030069710. (The application was later abandoned.)

An incorrect setting of the radiator radius is the biggest issue, but we can give some guidelines here. If the radius is taken as too small then the reconstruction is not going to be accurate. If the radius is too large then more terms will be required in the expansion; in other words, the values of the modal coefficients will not fall off as fast as they should. Hence the "correct" value of the radius is the value that causes the falloff of the modal coefficients to be the fastest.

We hope that the authors and the readers will find this additional information useful.

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Authors' Reply[4]

Our principal motivation for writing this paper was the discovery of the formula in our eq. (11), which expresses the on-axis pressure, both in the near field and in the far field, explicitly in terms of the expansion coefficients of the radially symmetric velocity profile with respect to the orthogonal Zernike circle polynomials of azimuthal order $m = 0$. With this formula in hand one is able to estimate the expansion coefficients—and thus the profile—from on-axis near-field pressure data. From this point onward one can proceed with estimating acoustical quantities, such as far field radiated power, reaction on radiator, and directivity. It is important to emphasize here that the pressure data are

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[1] R. M. Aarts and A. J. E. M. Janssen, *J. Audio Eng. Soc.*, vol. 57, pp. 1004–1015 (2009 Dec.).

[2] From an historical perspective it should be noted that sound field measurements and source reconstruction go back to the 1970s (acoustic holography, E. Williams) and the 1980s [G. Weinreich and E. B. Arnold, "Method for Measuring Acoustic Radiation Fields," *J. Acoust. Soc. Am.*, vol. 68, pp. 404–411 (1980 Aug.)].

[3] E. R. Geddes and L. Lee, *Audio Transducers* (GedLee Publishing, Novi, MI, 2001).

[4] Manuscript received 2010 March 9.

collected in the near field so that, in particular, no anechoic room is needed for the measurement. Thus the profile retrieval method combined with the forward computation of the far field generalizes Keele's method in the sense that nonuniform profiles are also allowed.

We agree with the commenter that there are a variety of sets of orthogonal functions on the disk which can be used to represent velocity profiles, and that one should be guided by particular practicalities in favoring one set over the other. The commenter has chosen in his book[3] the products of radial variable scaled Bessel functions and azimuthal integer-order cosines. This is a sound and renowned orthogonal set—the foundations of this method having been laid in the late 19th century (see Whittaker and Watson,[5] ch. XVII, exer. 20, p. 381) for the history of the method with regard to the radial functions—with practical merits for loudspeaker analysis, not in the least because of the validity of convenient Hankel transform relations for these functions. We regret that we did not yet compare the Zernike-based method systematically with other methods, including the one that the commenter has in chap. 4 of his book,[3] with respect to their merits for loudspeaker analysis. We would like to use this opportunity to touch on this subject now.

The Zernike circle polynomials $Z_n^m(\rho, \varphi) = R_{|m|+2n}^{|m|}(\rho) \exp(im\varphi)$, to be considered for $0 \leq \rho \leq 1$ and $0 \leq \varphi \leq 2\pi$, are used extensively in optical diffraction theory of aberrated systems. They arise naturally and uniquely among polynomial sets of orthogonal functions when certain forms of invariant requirements are made.[6,7] Furthermore there is the Hankel transform formula

$$\int_0^1 R_{|m|+2n}^{|m|}(\rho) J_m(u\rho)\rho \, d\rho = (-1)^n \frac{J_{2n+1}(u)}{u}. \qquad (1)$$

This formula renders the circle polynomials a very convenient tool both in optics, for computation of the point-spread function in the best-focus plane, and in acoustics, for computation of the far field of flexible, baffled-piston radiators in a similar manner as the commenter has this for products of Bessel functions and cosines in his book[3] (see chap. 4). The Hankel transform of the radial part $R_{|m|+2n}^{|m|}$ of the circle polynomial Z_n^m involves the Bessel function of order $2n + 1$, and is therefore, contrary to what the commenter states, quite different from the Hankel transform

$$\theta_n(s) = \frac{2sJ_1(s)}{s^2 - \beta_{0n}^2} \qquad (2)$$

which appears in eq. (4.2.12) and fig. 4.4 of the commenter's book. In particular, the right-hand side of Eq. (1) above has a zero of order $2n$ at $u = 0$ while the θ_n with $n \neq 0$ all have a zero of order 2 at $s = 0$. The Hankel transform in Eq. (1) is a very convenient starting point to express—directly or via King's formula—a variety of other acoustical quantities. This is done in our paper for the far field and in Aart's and Janssen[8] for quantities for which King's integral is instrumental. It is, at this moment, not obvious to us what could be done in this latter respect with the Hankel transforms of θ_n in Eq. (2) above.

Clearly the Bessel-times-cosine functions in the commenter's[3] chap. 4 are nonpolynomial in character, and so we should indicate why we have chosen in our paper to use the circle polynomials in a different manner than by referring to Wolf and Coworkers.[6,7] We are not aware of any analytic formulas for the on-axis pressure due to a nonuniform velocity profile that is expanded into orthogonal functions on the disk, other than our eq. (11) involving the expansion coefficients pertaining to the circle polynomials of azimuthal order $m = 0$. For our purposes the validity of such a formula, especially in the near field, is essential. The commenter's eq. (4.1.9-10) holds in the far field only, and would therefore not serve our purposes.

Both the system of Bessel-times-cosine functions and the system of Zernike circle polynomials can be used to represent velocity profiles and, in reverse direction, for estimating velocity profiles through expansion coefficients from far-field pressure data. For the Bessel-times-cosine functions the latter problem has been investigated by the commenter as he writes. An important issue here is the efficiency of a particular set of orthogonal functions in representing velocity profiles in terms of magnitude and number of required coefficients. There is an abundance of analytically given profiles with explicitly known expansion into circle polynomials (see Aart's and Janssen,[9] app. A). It is thus observed that for velocity profiles $v(\rho)$, such as $1 - \rho^2$, $\exp(-\alpha\rho^2)$, $\mathrm{sinc}(\alpha\rho)$, $2J_1(b\rho)/b\rho$, there is very fast coefficient decay, typically like $(B/n)^n$, with some B depending on the particular parameters considered in the profile. As an example we computed the normalized coefficients

$$\frac{\int_0^1 v(\rho) N_n(\rho)\rho \, d\rho}{\left(\int_0^1 N_n^2(\rho)\rho \, d\rho\right)^{1/2}}, \qquad n = 0, 1, \cdots \qquad (3)$$

of the velocity profile $v(\rho) = 1 - \rho^2$, $0 \leq \rho \leq 1$, for either choice

$$N_n(\rho) = J_0(\beta_{0n}\rho) \quad \text{or} \quad R_{2n}^0(\rho), \qquad n = 0, 1, \cdots. \qquad (4)$$

The numbers in Eq. (3) arise as coefficients when normalized N_n, rather than the N_n themselves, is used to expand v. It is thus found that the first choice in Eq. (4) yields

$$\frac{1}{2\sqrt{2}} \quad (n = 0), \qquad \frac{-2\sqrt{2}}{\beta_{0n}^2} \quad (n = 1, 2, \cdots) \qquad (5)$$

[5] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, MA, 1962, 4th ed.).

[6] A. B. Bhatia and E. Wolf, "On the Circle Polynomials of Zernike and Related Orthogonal Sets," *Proc. Camb. Phil. Soc.*, vol. 50, pp. 40–48 (1954).

[7] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, MA, 2002), chap. 9 and app. 7.

[8] R. M. Aarts and A. J. E. M. Janssen, "Sound Radiation Quantities Arising from a Resilient Circular Radiator," *J. Acoust. Soc. Am.*, vol. 126, pp. 1776–1787 (2009 Oct.).

[9] R. M. Aarts and A. J. E. M. Janssen, "On-Axis and Far-Field Sound Radiation from Resilient Flat and Dome-Shaped Radiators," *J. Acoust. Soc. Am.*, vol. 125, pp. 1444–1455, (2009 Mar.).

while the second choice in Eq. (4) yields

$$\frac{1}{2\sqrt{2}}, \quad \frac{-1}{2\sqrt{6}}, \qquad 0, 0, \cdots. \tag{6}$$

Thus the coefficients in Eq. (5) decay like $-2\sqrt{2}/\pi^2(n+1/4)^2$. While this decay is reasonably fast, it is definitely slower than the decay $(B/n)^n$ generally observed in the Zernike-based expansions for smooth profiles. To obtain the calculations for Eqs. (3) to (6) see Appendix.

We thank the commenter for sharing his experience with us regarding the practical issues that we raised in our paper. These comments are very useful and we shall bear them in mind when continuing our investigations. Here it should be noted, however, that the polar-plot far-field retrieval method, and the on-axis near-field retrieval method may behave quite differently for one or several of these issues.

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Appendix

The formulas in Eqs. (5) and (6) for the normalized expansion coefficients in Eq. (3) and either choice of N_n in Eq. (4) are developed in the following. Let us start with the first choice in Eq. (4). Since $\beta_{00} = 1$ and $J_0(1) = 1$, we readily find that the normalized coefficient for $n = 0$ equals $1/2\sqrt{2}$. Next let $\beta = \beta_{0n} > 0$ be such that $J_0'(\beta) = J_1(\beta) = 0$. We have by Abramowitz and Stegun,[10] 11.3.20, p. 484,

$$\int_0^z t^\nu J_{\nu-1}(t)\, dt = z^\nu J_\nu(z) \qquad \mathrm{Re}(\nu) > 0. \tag{7}$$

Hence by two partial integrations using Eq. (7) for $\nu = 0, 1$, we find

$$\int_0^1 (1-\rho^2) J_0(\beta\rho)\rho\, d\rho = \frac{2}{\beta^2} J_2(\beta). \tag{8}$$

Next, by Abramowitz and Stegun,[10] 11.4.5, p. 485, with $\nu = 0$, $m = n$, $\alpha_n = \beta_{0n}$, and $a = 0$, $b = 1$, we have

$$\int_0^1 |J_0(\beta_{0n}\rho)|^2 \rho\, d\rho = \frac{1}{2} J_0^2(\beta_{0n}). \tag{9}$$

Finally, from Abramowitz and Stegun,[10] 1st item in 9.1.27, p. 361, with $C = J$ and $\nu = 1$, we have

$$J_0(\beta_{0n}) + J_2(\beta_{0n}) = \frac{2}{\beta_{0n}} J_1(\beta_{0n}) = 0. \tag{10}$$

Hence from Eqs. (8) – (10) we get that

$$\frac{\int_0^1 (1-\rho^2) J_0(\beta_{0n}\rho)\rho\, d\rho}{\left(\int_0^1 J_0^2(\beta_{0n}\rho)\rho\, d\rho\right)^{1/2}} = \frac{-2\sqrt{2}}{\beta_{0n}^2} \tag{11}$$

as required. Note that $\beta_{0n} \approx (n + 1/4)\,\pi$ by Abramowitz and Stegun,[10] 9.5.12, p. 371 with $\nu = 1$ and $s = n$.

We finally consider the second choice in Eq. (4). We have

$$1 - \rho^2 = \frac{1}{2} - \frac{1}{2}(2\rho^2 - 1) = \frac{1}{2}R_0^0(\rho) - \frac{1}{2}R_2^0(\rho) + 0\cdot R_4^0(\rho) + 0\cdot R_6^0(\rho) + \cdots. \tag{12}$$

Then Eq. (6) follows from

$$\int_0^1 \left[R_{2n}^0(\rho)\right]^2 \rho\, d\rho = 1/2(2n+1).$$

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[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).